Universal finite-size scaling functions for percolation on three-dimensional lattices

Chai-Yu Lin
Institute of Physics, National Tsing Hua University, Hsinchu 30043, Taiwan

Chin-Kun Hu
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan

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Using a histogram Monte Carlo simulation method (HMCSM), Hu, Lin, and Chen found that bond and site percolation models on planar lattices have universal finite-size scaling functions for the existence probability E_p, the percolation probability P, and the probability W_n for the appearance of n percolating clusters in these models. In this paper we extend above study to percolation on three-dimensional lattices with various linear dimensions L. Using the HMCSM, we calculate the existence probability E_p and the percolation probability P for site and bond percolation on a simple-cubic (sc) lattice, and site percolation on body-centered-cubic and face-centered-cubic lattices; each lattice has the same linear dimension in three dimensions. Using the data of E_p and P in a percolation renormalization group method, we find that the critical exponents obtained are quite consistent with the universality of critical exponents. Using a small number of nonuniversal metric factors, we find that E_p and P have universal finite-size scaling functions. This implies that the critical E_p is a universal quantity, which is 0.265 ± 0.005 for free boundary conditions and 0.924 ± 0.005 for periodic boundary conditions. We also find that W_n for site and bond percolation on sc lattices have universal finite-size scaling functions. [S1063-651X(98)09008-4]

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I. INTRODUCTION

Percolation is related to many interesting problems in physics [1]. In recent years, there has been a number of investigations concerning universality and scaling in percolation problems. One of the research objects is the existence probability $E_p(L,p)$ [2–4], which is the probability that percolating clusters exist on a lattice G with a linear dimension L and a site or bond occupation probability p. E_p was called the spanning probability by Ziff [5], and the crossing probability by Langlands and co-workers [6,7]. A mathematical definition of E_p will be given in Sec. II.

First, from the self-duality argument, for bond percolation on $L \times L$ square (sq) lattices with free boundary conditions and spanning rule R_1 defined by Reynolds, Stanley, and Klein [8] (i.e., checking percolation in one direction only), we can directly see that $E_p(p_c,L)=0.5$ at the critical point $p_c=0.5$ [5]. This result is consistent with the fixed-point equation of the one-parameter renormalization group (RG) transformation, i.e. $E_p(L,p_c)=p_c$. However, Ziff [5] and Grassberger [9] found that $E_p(\infty,p_c)=0.5 \neq p_c \approx 0.592746$ for site percolation on $L \times L$ sq lattices as $L \rightarrow \infty$. Such numerical result contradicts the fixed-point equation of the one-parameter RG transformation [5], i.e., $E_p(L,p_c)=p_c$ is not satisfied, but confirms the universality of E_p at the critical point [5]. Hu pointed out that the fixed-point equation of the cell-to-cell RG transformation gives the correct critical E_p [4], and used a histogram Monte Carlo simulation method (HMCSM) (Refs. [2,4]) to confirm this idea. Aharony and Hovi used another RG argument to resolve the apparent contradiction [10]. Sahimi and Rassamdana [11] and Hu, Chen, and Wu [12] discussed the convergence of fixed points of a series of different RG equations. Hu, Chen, and Wu [12] found that the critical points determined by cell-to-cell RG transformations converge to their final value faster than those determined by cell-to-site RG transformations.

In 1992, Langlands, Pichet, Pouliot, and Saint-Aubin (LPPS) [6] investigated site and bond percolations on sq, honeycomb (hc) and triangular (tr) lattices with rectangular domains. They proposed that when the aspect ratios of sq, hc, and tr lattices are a, $a\sqrt{3}$, and $a\sqrt{3}/2$, respectively, then $E_p(\infty,p_c)$ on these lattices is a universal function of a. Cardy derived an exact formula for critical E_p as a function of a by conformal field theory [13]. The agreement between Cardy's formula and LPPS's numerical results is excellent.

In addition to calculating the physical quantities at critical points, Hu, Chen, and Lin (HLC) used the HMCSM [2,3] to calculate finite-size scaling functions for E_p and the percolation probability P. They found that such finite-size scaling functions depend sensitively on the boundary conditions [4], spanning rules [14], and aspect ratios of the lattice [15,16]. Using the relative aspect ratios proposed by LPPS and a small number of nonuniversal metric factors [17], Hu, Lin, and Chen [18,19] obtained universal finite-size scaling functions for E_p and P of site and bond percolation on sq, hc, and tr lattices. Finite-size corrections to the universal finite-size scaling functions were discussed by Aharony and Hovi [10,20].

Another geometrical quantity which is interesting and not well studied is the probability for the appearance of n percolating clusters, W_n [21]. Using the HMCSM, Hu found that W_n has very good scaling behavior, and that the finite-size scaling functions for W_n depend sensitively on boundary conditions of the lattice [21]. Using the HMCSM and nonuniversal metric factors obtained by HLC [18], Hu and Lin (HL) found that W_n for site and bond percolations on finite sq, hc, and tr lattices fall on the same universal finite-size scaling functions, which show many interesting behaviors as the aspect ratio of the lattice increases [22]. In two-dimensional lattices, one might expect that there exists only a single percolating cluster at the critical point [23,24]. However, HL [22] found that there is a nonzero probability that the system has multiple percolating clusters. Sen [25] confirmed this result. Using nonuniversal metric factors, Hu and Wang found that continuum percolation of soft and hard disks have the same universal finite-size scaling functions as lattice percolation [26]. In rectangular domains, Monetti and Albano [27] also studied the dependence of number of percolating clusters with the aspect ratio R when $R \gg 1$. The values of W_n at the critical point are useful for understanding $\sigma_{xx}^{\max}$ in quantum Hall effects [28–30].

In recent years, there have been both mathematical and computational studies of W_n. A recent review was given by Stauffer [31]. Aizenman [32] derived upper and lower bounds of W_n for two-dimensional percolation at the critical point, which was confirmed by the Monte Carlo results of Shchur and Kosyakov [33]. Using conformal theory, Cardy [34] proposed an exact formula for critical W_n for large aspect ratios.

Almost all of the results mentioned above are for two-dimensional systems. However, many interesting and important problems are in three-dimensional space, where exact solutions are almost impossible, and one must use approximate methods to study the problem. Since numerical computations require a lot of memory and computing time, progress in the numerical studies of three-dimensional percolation has been slower than that in two-dimensional percolation. There are some studies of percolation in high-dimensional space [35–44], but there is still no study of the universal finite-size scaling functions for percolation in three-dimensional space. Using the HMCSM [2–4], in this paper we study the universal finite-size scaling functions for bond and site percolation on three-dimensional lattices.

This paper is organized as follows. The numerical technique of histogram Monte Carlo simulation method [2,3], and related formulas for the calculation of critical exponents and finite-size scaling functions are briefly reviewed in Sec. II. The calculated results for the existence probability E_p, the percolation probability P, and the number of percolation clusters W_n are presented in Sec. III. Finally, some related theoretical problems are discussed in Sec. IV.

II. COMPUTATIONAL ALGORITHM AND THEORETICAL FORMULATION

The HMCSM proposed by Hu [2,3] is useful for calculating E_p and P. Here we briefly review the HMCSM for site percolation [4]. The extension to bond percolation [2,15] is straightforward.

In site percolation on a d-dimensional lattice G of N sites, each site of G is occupied with a probability p, where $0 \leqslant p \leqslant 1$. A cluster which extends from one side of G to the opposite side of G is a percolating cluster. A subgraph which contains at least one percolating cluster is a percolating subgraph and denoted by G'_p. Then we have the definitions

$$E_p(L,p) = \sum_{G'_p \subseteq G} p^{v(G'_p)}(1-p)^{N-v(G'_p)}, \qquad (1)$$

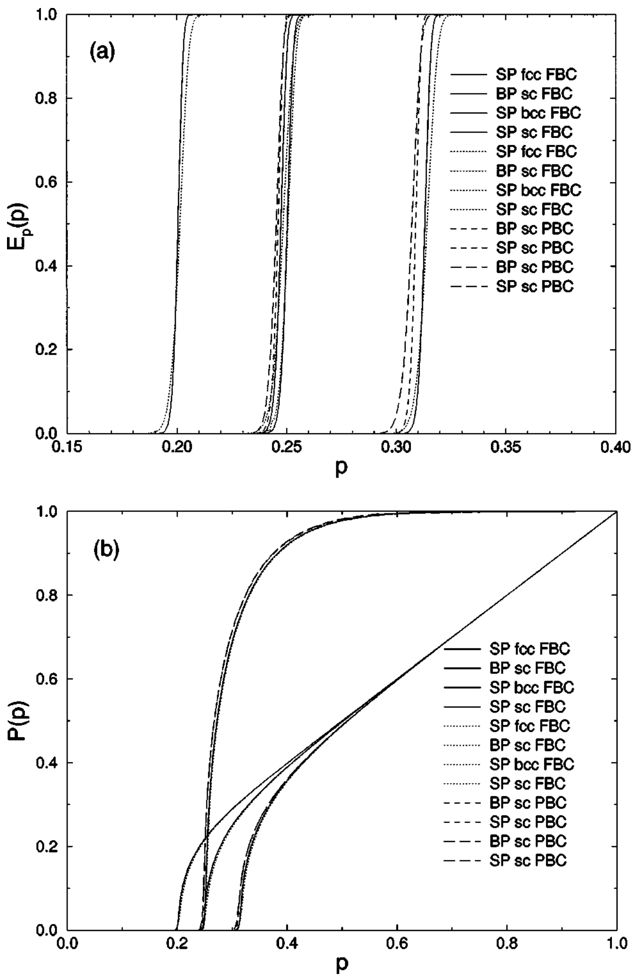


FIG. 1. Results for site percolation (SP) on 128^3 and 80^3 simple-cubic (sc), body-centered-cubic (bcc) and face-centered-cubic (fcc) lattices, and bond percolation (BP) on 80^3 and 64^3 sc lattices. The solid (dotted) lines from left to right are for SP on $128^3(80^3)$ fcc, BP on $80^3(64^3)$ sc, SP on $128^3(80^3)$ bcc, and SP on $128^3(80^3)$ sc lattices with fbc. The dashed (dot-dashed) lines from left to right are for BP on $80^3(64^3)$ sc and SP on $128^3(80^3)$ sc lattices with pbc. (a) E_p as a function of p. (b) P as a function of p.

$$P(L,p) = \sum_{G'_p \subseteq G} p^{v(G'_p)}(1-p)^{N-v(G'_p)} N^*(G'_p)/N, \qquad (2)$$

where $v(G'_p)$ is the number of occupied sites in G'_p. The summations in Eqs. (1) and (2) are over all percolating subgraphs G'_p of G, and $N^*(G'_p)$ is the total number of sites in the percolating cluster of G'_p. In the HMCSM, we choose w different values of p. For a given $p = p_j$, $1 \leqslant j \leqslant w$, we generate N_R different subgraphs G'. The data obtained from the wN_R different G' are then used to construct three arrays of numbers of length N with elements $N_p(v)$, $N_f(v)$, and $N_{pp}(v)$, $0 \leqslant v \leqslant N$, which are, respectively, the total numbers of percolating subgraphs with v occupied sites, nonpercolating subgraphs with v occupied sites, and the sum of $N^*(G'_p)$ over percolating subgraphs with v occupied sites. After a large number of simulations, the existence probability E_p and the percolation probability P at any value of the site occupation probability p can then be calculated approximately from the equations [2–4]

TABLE I. Critical points and exponents for site and bond percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices. Here L_1 and L_2 respresent linear dimensions of lattices in cell-to-cell RG transformations, and only lattices with free boundary conditions are considered. The data inside parentheses are included for comparison with our results. The critical points p_c for site and bond percolation on sc lattice are taken from Refs. [35] and [43], respectively; p_c for site percolation on bcc and fcc lattices are taken from Ref. [37]; y_t and y_h inside parentheses are calculated from the data of Ref. [43].

Percolation	site	site	site	bond
Lattice	sc	bcc	fcc	sc
$L_1 \leftrightarrow L_2$	$128 \leftrightarrow 80$	$128 \leftrightarrow 80$	$128 \leftrightarrow 80$	$100 \leftrightarrow 80$
p_c	0.3116 ± 0.0001	0.2459 ± 0.0001	0.1992 ± 0.0001	$0.24887 \pm 0.000\,06$
	$(0.311605 \pm 0.000\,010)$	(0.2464 ± 0.0007)	(0.1998 ± 0.0006)	$(0.248\,812\,6 \pm 0.000\,000\,5)$
y_t	1.12 ± 0.02	1.12 ± 0.03	1.15 ± 0.02	1.13 ± 0.03
	(1.123 ± 0.025)			
y_h	2.49 ± 0.01	2.46 ± 0.01	2.47 ± 0.01	2.52 ± 0.02
	(2.523 ± 0.004)			

$$E_p(L,p) = \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^N \frac{N_p(v)}{N_p(v)+N_f(v)}, \tag{3}$$

$$P(L,p) = \frac{1}{N} \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^N \frac{N_{pp}(v)}{N_p(v)+N_f(v)}, \tag{4}$$

where $C_v^N = N!/(N-v)!v!$.

This method was generalized to evaluate the probability of the appearance of n percolating clusters, W_n [21,22]. A percolating subgraph with n percolating clusters is denoted by G_n'. Now we have the definition

$$W_n(L,p) = \sum_{G_n' \subseteq G} p^{v(G_n')} (1-p)^{N-v(G_n')}. \tag{5}$$

By the same procedure, W_n can be calculated approximately from the equation

$$W_n(L,p) = \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^N \frac{N_{pn}(v)}{N_p(v)+N_f(v)}, \tag{6}$$

where $N_{pn}(v)$ is the number of percolating subgraphs with n percolating clusters and v occupied sites. It is obvious that $E_p = \Sigma_{n=1}^{\infty} W_n$ and $N_p(v) = \Sigma_{n=1}^{\infty} N_{pn}(v)$.

The percolation renormalization group transformation from lattice G_1 of linear dimension L_1 to lattice G_2 of linear dimension L_2, where $L_1 > L_2$, is given by the equation

$$E_p(L_2,p') = E_p(L_1,p), \tag{7}$$

which gives the renormalized site probability p' as a function of p. The fixed point of Eq. (7) gives the critical point p_c. The thermal scaling power y_t and the field scaling power y_h, which is equal to the fractal dimension D of the percolating cluster at p_c, can be obtained from the equations

$$y_t = \frac{1}{\nu} = \frac{\ln\left(\frac{\partial p'}{\partial p}\right)_{p_c}}{\ln\frac{L_1}{L_2}}, \quad y_h = D = \frac{\ln\frac{P(L_1,p_c)L_1^d}{P(L_2,p_c)L_2^d}}{\ln\frac{L_1}{L_2}}. \tag{8}$$

Let us consider a system of linear dimension L near the critical point. According to the theory of finite-size scaling [1,45,46], if the dependence of a physical quantity Q of a thermodynamic system on a parameter t, which vanishes at the critical point $t=0$, is of the form $Q(t) \sim t^a$ near the critical point, then the corresponding quality $Q(t,L)$ is of the form

$$Q(t,L) \sim L^{-ay_t} F(tL^{y_t}), \tag{9}$$

where $y_t (=\nu^{-1})$ is the thermal scaling power, ν is the correlation length exponent and $F(x) (x=tL^{y_t})$ is the finite-size scaling function. It follows from Eq. (9) that the scaled data $Q(t,L)L^{ay_t}$ for different values of L and t are described by a single function $F(x)$.

Although different systems with the same spatial dimensionality and the same symmetry properties have the same set of critical exponents, it is widely believed that different lattices have different finite-size scaling functions. In 1984, Privman and Fisher [17] proposed the concept of a universal finite-size scaling function and nonuniversal metric factors. In particular, they proposed that, near $t=0$, the singular part of the free energy can be written as

$$f_s(t,L) \sim L^{-d} Y(CtL^{y_t}), \tag{10}$$

where d is the spatial dimensionality of the lattice, Y is a universal scaling function, and C is a nonuniversal metric factor.

Now we consider universal finite-size scaling functions for the existence probability E_p and the percolation probability P. In the limit $L \to \infty$, E_p approaches the step function $\Theta(p-p_c)$; if we write $E_p \sim (p-p_c)^a$ for $p > p_c$, then the critical exponent a is 0. $P(p,L)$ is the fraction of lattice sites in the percolating cluster, and is the order parameter of the system. In the limit $L \to \infty$, $P \sim (p-p_c)^\beta$ for $p > p_c$. According to Eq. (9), we may write $E_p = F_1(z)$ and $P(z)L^{\beta y_t} = F_2(z)$, where $z = (p-p_c)L^{y_t}$ is a scaling variable and F_1 and F_2 are finite-size scaling functions. To find universal finite-size scaling functions, we introduce the nonuniversal metric factors D_1, D_2, and D_3, as in the paper by HLC [18,19] and consider E_p as a function of $x_1 = D_1 z$ and $D_3 P L^{\beta y_t}$ as a function of $x_2 = D_2 z$. The *universal* finite-size scaling functions for E_p and P are denoted by F and S, respectively.

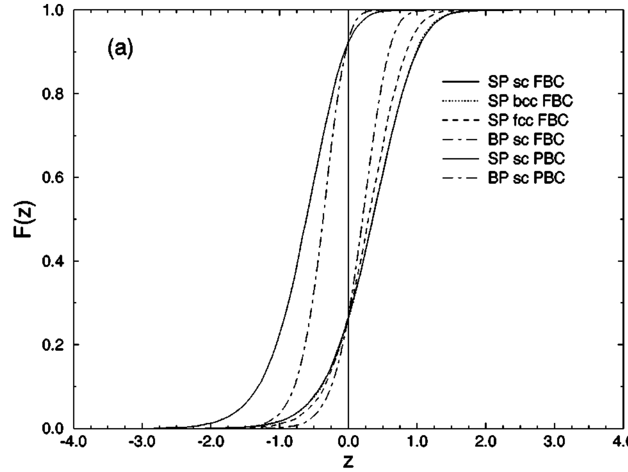
III. NUMERICAL RESULTS

Typical calculated results of E_p and P for site percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices, and bond percolation on sc lattices with both free boundary conditions (FBC's), and periodic boundary conditions (PBE's) are shown in Figs. 1(a) and 1(b), respectively. Our periodic boundary conditions are periodic in three directions, similar to the case of planar lattices considered by Hu and co-workers [18,47].

Since there are no exact solutions for p_c, y_t, and y_h for percolation on three-dimensional lattices, we use Eq. (7) and (8) to obtain approximate numerical values. For site percolation, we use $L_1 = 128$ and $L_2 = 80$. For bond percolation, we use $L_1 = 100$ and $L_2 = 80$, which are larger than those used in Fig. 1, so that we may obtain accurate p_c, y_t, and y_h. The calculated results are shown in Table I, in which results obtained by other methods [35,37,43] are also shown for comparison. Data shown in Table I support the idea that critical exponents for percolation on lattices of the same dimensionality are universal [1].

In [35], Ziff and Stell found that for site percolation on a sc lattice the critical point is $p_c(\mathrm{sc}) = 0.311\,605 \pm 0.000\,010$. In a very recent paper [43], Lorenz and Ziff had a very precise determination of critical exponents for percolation on three-dimensional lattices. They found that the Fisher exponent τ is 2.189 ± 0.002 and the scaling function exponent σ is 0.445 ± 0.01. From these data and scaling relations [1], we find $y_t = 1/\nu = 1.123 \pm 0.025$ and $y_h = D = 2.523 \pm 0.004$. For site percolation on sc lattices, we use $N_R = 55\,000$ for $L_1 = 128$, and $N_R = 70\,000$ for $L_2 = 80$, and $w = 345$ for both cases to obtain $p_c(\mathrm{sc}) = 0.3116 \pm 0.0001$, $y_t = 1.12 \pm 0.02$, and $y_h = 2.49 \pm 0.01$, which are very close to the result of Refs. [35] and [43]. In 1995, Hu [42] used the same procedure to calculate the same qualities by using $L_1 = 80$ and $L_2 = 64$. The values found here are closer to the results of Refs. [35] and [43]. This may be related to the fact that now we use larger lattices, and the finite-size correction is smaller.

Using the data of Fig. 1 and $p_c = 0.3116$, 0.2459, 0.1992, and 0.2488 for site percolation on sc, bcc, and fcc lattices, and bond percolation on a sc lattice, respectively, and $y_t = 1.123$ and $y_h = 2.523$, we plot E_p and $PL^{\beta y_t}$ as a function of $z = (p-p_c)L^{y_t}$ in Figs. 2(a) and 2(b), respectively, in which the scaling functions are denoted by $F(z)$ and $S(z)$, respectively. The 12 curves of Figs. 1(a) and 1(b) collapse nicely into six curves in Figs. 2(a) and 2(b), i.e., they have good finite-size scaling behavior. It is of interest to note that curves of different models with the same boundary conditions go through the same point at $z = 0$. This verifies the universality of critical E_p for the same boundary conditions [44], and provides a good basis to study universal finite-size scaling functions (UFSSF's).



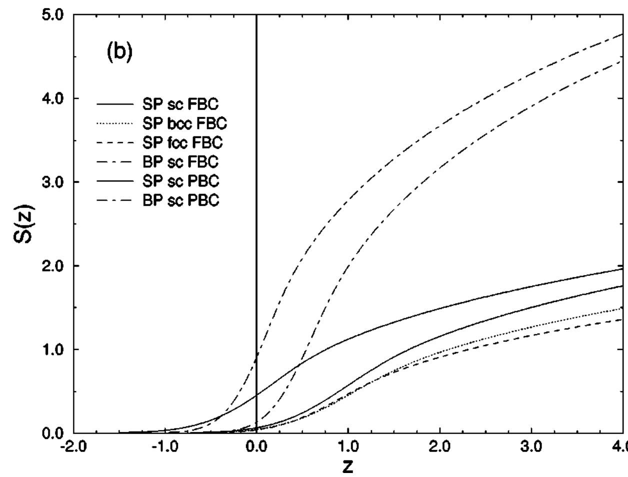


FIG. 2. Scaling functions for site percolation (SP) on sc, bcc, and fcc lattices, and bond percolation (BP) on sc lattices. The data are taken from Fig. 1. (a) F as a function of $z = (p-p_c)L^{y_t}$. The slopes of the solid (dotted) lines at $z = 0$ from small to large are for SP on $128^3(80^3)$ sc, SP on $128^3(80^3)$ bcc, SP on $128^3(80^3)$ fcc, and BP on $80^3(64^3)$ sc lattices with fbc. The slopes of the dashed (dot-dashed) lines at $z = 0$ from small to large are for SP on $128^3(80^3)$ sc and BP on $80^3(64^3)$ sc lattices with pbc. (b) S as a function of $z = (p-p_c)L^{y_t}$. The values of the solid (dotted) lines at $z = 0$ from small to large are for SP on $128^3(80^3)$ fcc, SP on $128^3(80^3)$ bcc, SP on $128^3(80^3)$ sc, and BP on $80^3(64^3)$ sc lattices with fbc. The values of the dashed (dot-dashed) lines at $z = 0$ from small to large are for SP on $128^3(80^3)$ sc and BP on $80^3(64^3)$ sc lattices with pbc.

To study UFSSF's, we used the application program *xvgr* to fit data of Figs. 2(a) and 2(b) as polynomials in z. The coefficients of the linear terms for $F(z)$ are used to calculate D_1, and the coefficients of the constant and linear terms for $S(z)$ are used to calculate D_3 and D_2, respectively. We defined D_1, D_2 and D_3 to be 1 for site percolation on sc lattices, and used this definition to calculate D_1, D_2, and D_3 for other models. The calculated results are listed in Table II, where the values for periodic boundary conditions are represented by D_1', D_2', and D_3'. We then plot $E_p(p,L)$ as a function of $x = D_1(p-p_c)L^{y_t} = D_1 z$ in Fig. 3(a), and $D_3 P(p,L) L^{\beta y_t}$ as a function of $x = D_2(p-p_c)L^{y_t} = D_2 z$ in

TABLE II. Nonuniversal metric factors for site and bond percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices. The values of w and N_R used in the simulations are also shown. w, N_R, D_1, D_2, and D_3 are for lattices with free boundary conditions; w', N_R', D_1', D_2', and D_3' are for lattices with periodic boundary conditions.

Percolation	site	site	site	bond
Lattice	sc	bcc	fcc	sc
w	345	345	345	345
N_R	55 000	30 000	20 000	25 000
D_1	1	1.004 ± 0.012	1.156 ± 0.017	1.671 ± 0.025
D_2	1	1.037 ± 0.037	1.194 ± 0.045	1.678 ± 0.038
D_3	1	1.283 ± 0.029	1.485 ± 0.036	0.503 ± 0.032
w	345	-	-	345
N_R	30 000	-	-	15 000
D_1'	1	-	-	1.701 ± 0.031
D_2'	1	-	-	1.699 ± 0.043
D_3'	1	-	-	0.502 ± 0.020

Fig. 3(b). Figures 3(a) and 3(b) show that E_p and P possess well-defined UFSSF's, which are denoted by $F(x)$ and $S(x)$ for E_p and P, respectively.

It is of interest to note that for each column in Table II, D_1 is consistent with D_2 within numerical uncertainty, and for bond percolation on sc lattices the values of D_1, D_2, and D_3 for free boundary conditions are consistent with those for periodic boundary conditions. In other words, as in the case of percolation on planar lattices [18,19], only a small number of nonuniversal scaling metric factors are needed to reach the universal scaling functions shown in Figs. 3(a) and 3(b). We find that $E_p(p_c,L)=F(0)$ of Fig. 3(a) for free boundary conditions is equal to 0.265 ± 0.005 [44], which is quite different from the result $E_p(p_c,\infty)\approx0.42$ obtained in Ref. [39], but is consistent with critical E_p for continuum percolation of soft spheres and hard spheres in three-dimensional space with free boundary conditions [48]. For periodic boundary conditions, we find that $E_p(p_c,L)=F(0)=0.924\pm0.005$.

To study the scaling behavior of W_n, we use Eq. (8) to evaluate W_n for site percolation on $128\times128\times64$, $100\times100\times50$, $80\times80\times40$, and $64\times64\times32$ sc lattices with free boundary conditions. The calculated results as a function of p and as a function of $z=(p-p_c)L^{y_t}$ are shown in Figs. 4(a) and 4(b), respectively, where $W_0=1-E_p$. Figure 4(b) shows that W_n has a reasonably good scaling behavior. However, it is not as good as that found for bond percolation on square lattices [22]. We consider that there are several possible reasons. (1) In the present paper, we do not have exact p_c, y_t, and y_h, while in Ref. [22] there were exact p_c, y_t, and y_h for bond percolation on sq lattices. (2) The finite-size scaling correction for three-dimensional systems is larger than that in Ref. [22], thus we may need to do simulations on larger lattices in order to obtain better scaling behavior.

To study the UFSSF for W_n, we calculated $W_n(L_1,L_2,L_3,p)$ for site percolation on an $80\times80\times80$ sc lattice, and for bond percolation on a $64\times64\times64$ sc lattice with free boundary conditions. The calculated W_n as a function of $x=D_1(p-p_c)L^{y_t}$ are shown in Fig. 5, where D_1 is taken from the last column of Table II. Figure 5 shows that all calculated results for each n fall on the same universal scaling function, $U_n(x)$. Sen [25] found that the probability of getting more than one percolating cluster at p_c for site percolation on sc lattices is about 0.014, which is quite consistent with our result: $U_2(0)\approx0.013$, and $U_n(0)$ is vanishing small for $n>2$.

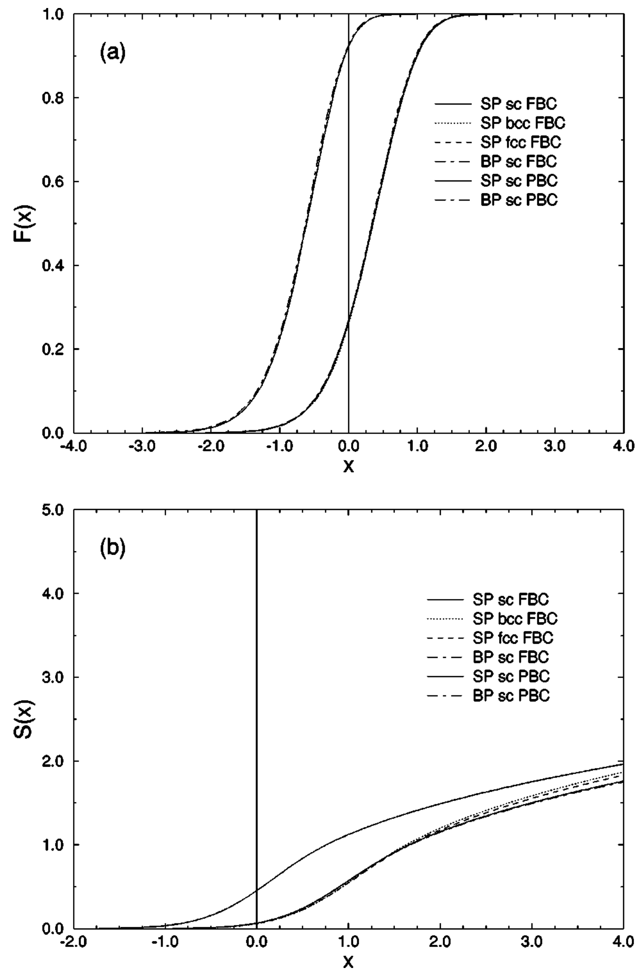


FIG. 3. (a) The calculated E_p for the site percolation on sc, bcc, and fcc lattices and bond percolation on sc lattices, where $x=D_1(p-p_c)L^{y_t}=D_1z$. The scaling function is $F(x)$. The lower (upper) curves are for free (periodic) boundary conditions. (b) The calculated $D_3PL^{\beta y_t}$ for the site percolation on sc, bcc, and fcc lattices and bond percolation on sc lattices as a function of x, where $x=D_2(p-p_c)L^{y_t}=D_2z$. The scaling function is $S(x)$. The lower (upper) curves are for free (periodic) boundary conditions.

IV. DISCUSSION

In Ref. [4], Hu found that finite-size scaling functions for percolation on square lattices depend sensitively on boundary conditions of the lattice. In particular, at $x=0$, $F=0.50$ for free boundary conditions (fbc), and $F=0.93$ for periodic boundary conditions (pbc). In the present paper, we find that

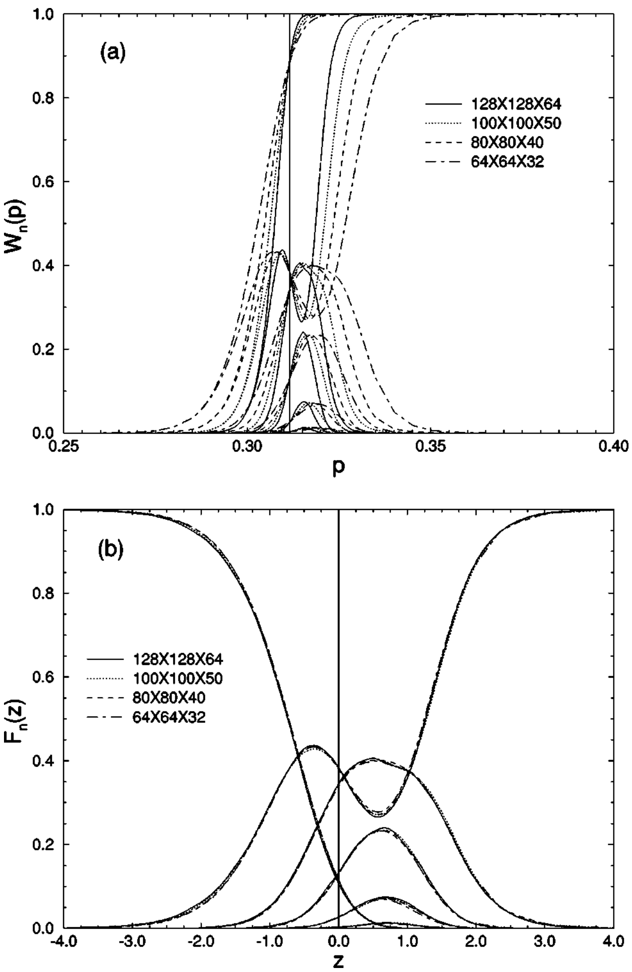


FIG. 4. (a) $W_n(L_1,L_2,L_3,p)$ for site percolation with free boundary conditions on $128\times128\times64$, $100\times100\times50$, $80\times80\times40$, and $64\times64\times32$ sc lattices, which are represented by solid, dotted, dashed, and dot-dashed lines, respectively. (b) The data of (a) are plotted as a function of $z=(p-p_c)L^{y_t}$. The scaling functions for W_n are denoted by $F_n(z)$. The monotonic decreasing function is for F_0. The S shaped curve is for F_1. The bell shaped curves from top to bottom are for $f_n(z)$, with n being 2, 3, 4 and 5, respectively.

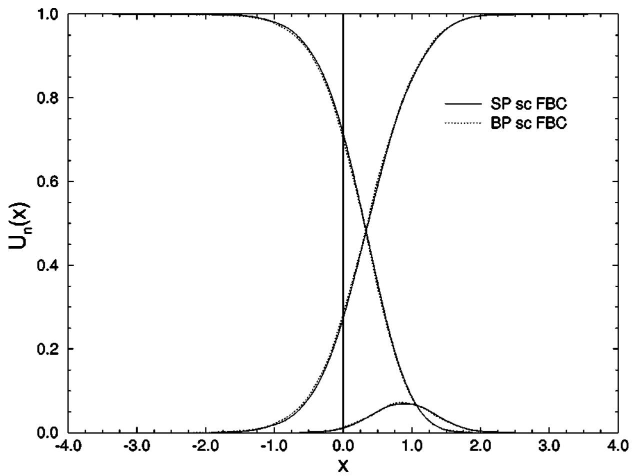


FIG. 5. The universal finite-size scaling function $U_n(x)$, for W_n on the L^3 system. It is plotted as a function of scaling variable $x=D_1(p-p_c)L^{y_t}$, where D_1 is taken from Table II. The solid and dot-dashed lines respresent, respectively, site and bond percolation on sc lattices with free boundary conditions. Here only results for $n=0$, 1, and 2 are shown. The monotonic decreasing (increasing) function is for $F_0(F_1)$. The bell shaped curve is for F_2.

finite-size scaling functions for percolation on three-dimensional lattices, e.g., $F(x)$ of Fig. 3(a), also depend sensitively on boundary conditions of the lattice. In particular, at $x=0$, $F=0.265\pm0.005$ for fbc and $F=0.924\pm0.005$ for pbc. It is of interest to note that as the spatial dimensionality increases, the difference between the values of F at $x=0$, e.g. critical E_p, for pbc and fbc also increases. If this trend continues, we may predict that for four- and higher-dimensional lattices, the difference of critical E_p for pbc and fbc would be larger than 0.659.

When we used the histogram Monte Carlo renormalization group method to calculate thermal scaling power y_t and fractal dimension D for percolation on planar lattices [2,4], we found that lattices of medium size can give very accurate y_t, and we should use much larger lattices in order to obtain a D of comparable accuracy. We have a similar experience when we calculate y_t and D for three-dimensional lattices. If we increase the lattice sizes, we can increase the accuracy of D shown in Table I.

In Refs. [18,22], we found universal finite-size scaling functions for E_p, P, and W_n of bond and site percolation on planar lattices. In the present paper, we find that the results for percolation on planar lattices may be extended to percolation on three-dimensional lattices.

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